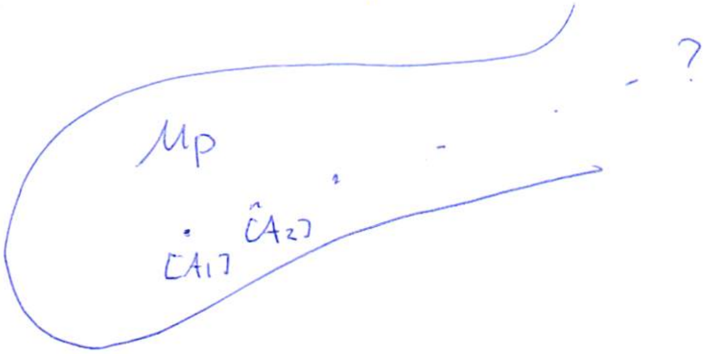
§8 Uhlenbeck compactness.



<u>Thm 1</u> (Uhlenbeck)

X: **closed**, oriented smooth 4-manifold.

$P \to X$, principal G-bundle over X

Let's take $G = SU(2)$)

Let $\{A_n\}$: sequence of L^2_2-ASD connections on P

Then after taking a subsequence,

(1) $\exists \{p_1, \cdots, p_k\} \subset X$, finite set of points

(2) $\exists \{m_1, \cdots, m_k\} \subset \mathbb{Z}_{>0}$

(3) $\exists A_\infty$: L^2_2 - ASD connection on a smooth $SU(2)$-bundle $\exists P'$ over X

with $c_2(P') \lneq c_2(P)$

$$\left(c_2(P) = c_2(P') + \sum_{j=1}^{k} m_j \right)$$

(4) $\exists L^2_3$ - gauge transformations

$$g_n : P'|_{X \setminus \cup p_j} \xrightarrow{\cong} P|_{X \setminus \cup p_j}$$

such that

(i) for $\forall K \underset{cpt}{\subset} X \setminus \cup p_j$

$$g_n(A_n|_K) \to A_\infty|_K \text{ in } L^2_2$$

(ii) $|F_{A_n}|^2 dvol_g \to |F_{A_\infty}|^2 dvol_g + \sum_j m_j \delta_{p_j}$

as measures on X in the weak-$*$ topology,

where δ_{p_j} is the δ-measure (of total mass $8\pi^2$)

<u>weak-$*$ convergence</u>

X : normed space

$\{f_n\} \subset X'$: dual of X

For each $x \in X$, $\{f_n\}$ converges to $f_\infty \in X'$

in the weak-$*$ topology

$\overset{i}{\iff} \lim_{n \to \infty} f_n(x) = f_\infty(x)$ for $\forall x \in X$

Thm (Uhlenbeck gauge)

Let B^4 : 4-ball of radius 1, and consider the trivial bundle on it. Fix a reference metric, say A_0

Then $\exists \varepsilon > 0$, $\exists C > 0$, $\forall A$: connection on $\overline{B^4}$, if $\|F_A\|_{L^2} < \varepsilon$, then

$\exists g$: L^2_3 - gauge transformation such that

(1) $d_{A_0}^* (g(A) - A_0) = 0$

(2) $\lim_{|x| \to 1} g(A_r) = 0$, A_r: the radial component of A

(3) $\| g(A) - A_0 \|_{L^2_2} \leq C \|F_A\|_{L^2}$

Using this, one can prove:

Thm (Regularity)

A : ASD connection on the trivial bundle over a 4-ball B^4_1 of radius 1

Suppose $\|F_A\|_{L^2(B_1)} < \varepsilon$

then $\exists g \in \mathcal{G}_{L^2_3}$ on $B^4_{1/2}$ such that

$$\| g(A) \|_{C^k(B_{1/2})} \leq C(k) \|F_A\|_{L^2(B_1)}$$

In particular, $\max_{B^4_{1/2}} |F_A| \leq C \|F_A\|_{L^2(B_1)}$

Thm (Removal singularity)

A : ASD connection on a principal G-bundle P over a punctured 4-ball, say $B^4 \setminus \{0\}$

with $\begin{cases} L^2_2 \text{ over } \forall K \underset{\text{compact}}{\subset} B^4 \setminus \{0\} \\ \|F_A\|_{L^2(B^4)} < \infty \end{cases}$

Then $\exists P'$: principal G-bundle over B^4, extension of P

and $\exists A'$: ASD L^2_2-connection on P', extension of A.

Sketch of proof of Thm 1

Fix $\varepsilon > 0$ in Thm (Uhlenbeck gauge)

- First $\{ |F_{A_n}|^2 dvol_g \}$ has a converging subsequence to a measure μ in the weak-* topology.

as it has the constant L^1-norm $8\pi^2 c_2(P)$

Consider
$$S := \left\{ x \in X \;\middle|\; \begin{matrix} \mu \text{ has a } \delta\text{-mass} \\ \text{of norm at least } \varepsilon/2 \end{matrix} \right\}$$

Since the norm of μ is $8\pi^2 c_2(P)$, and X is compact S is a finite set of points

If $x \notin S$, after taking a subsequence

$\exists\, U$: sufficiently small ball containing x

$\exists\, g_n$: L^2_3-gauge transformations on $P|_U$.

Such that
$$g_n(A_n) \longrightarrow A_\infty \quad \text{in } L^2_2(U)$$

Then cover $X \setminus S$ by countably-many such balls U and take a diagonal subsequence.

patch together the local gauge transformations

to global gauge transformations g_n on $X \setminus S$

Then $\exists \hat{P}$ over $X \setminus S$

$\exists$ bundle isomorphisms

$$\hat{g_n} : \hat{P} \to P|_{X \setminus S}$$

$\exists \hat{A}$: ASD connections on $\hat{P}$ such that

$$\hat{g_n}(A_n) \longrightarrow \hat{A} \text{ in } L^2_2$$

on $\forall K \underset{\text{compact}}{\subset} X \setminus S$

Finally by removal singularity theorem,

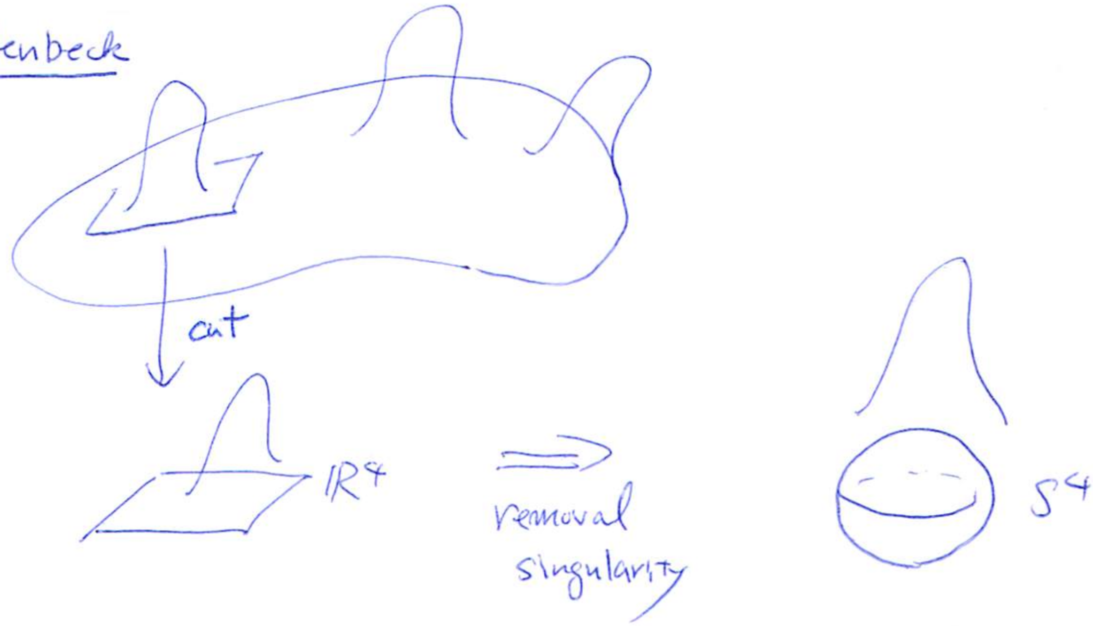
extend $\begin{cases} \hat{P} \text{ to } P' \text{ over } X \\ \hat{A} \text{ to } A' : \text{ASD connection on } P' \end{cases}$

Rmk

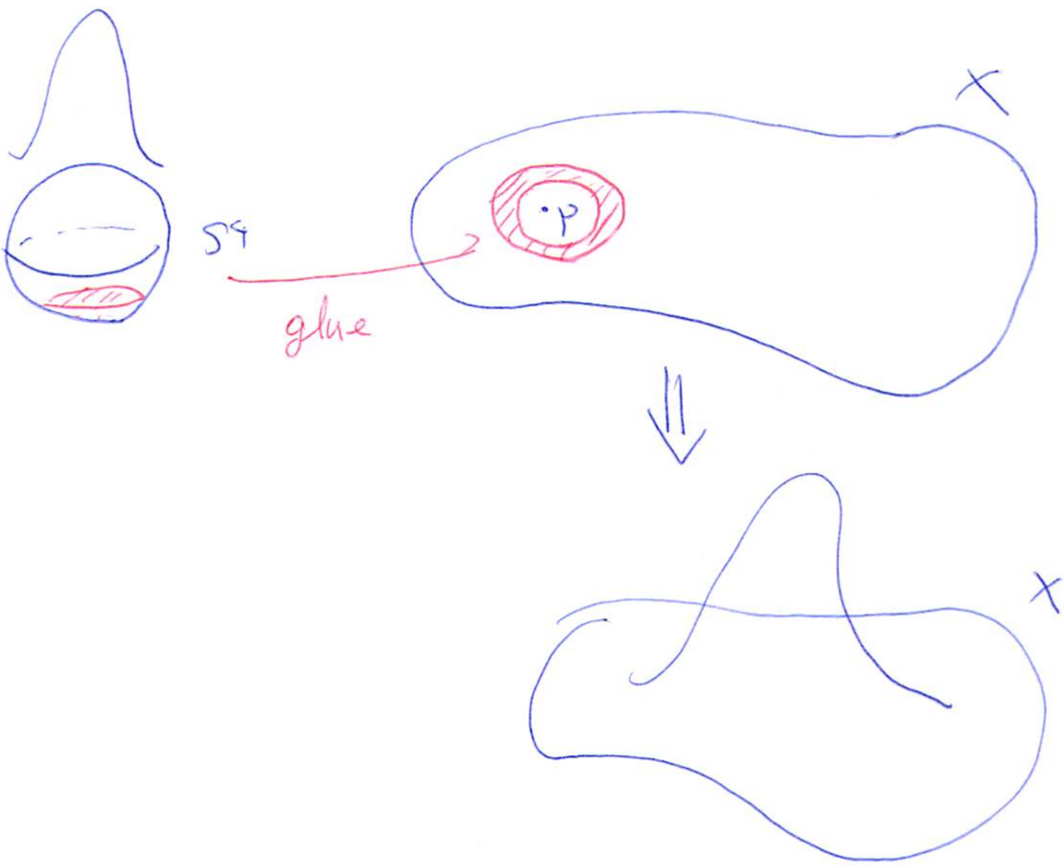
$$0 \leq c_2(P') \leq c_2(P)$$

§9 Taubes' gluing

Uhlenbeck



Taubes



Thm (Taubes)

Let $\begin{cases} X : \text{closed, oriented, smooth 4-manifold} \\ \quad \text{with } b^+(X) = 0 \\ P \to X : \text{principal } SU(2)\text{-bundle over } X \\ \text{with } c_2(P) > 0 \end{cases}$

Then P admits smooth irreducible ASD connections.

Rmk

Can drop the assumption $b^+(X) = 0$,

but need $c_2(P) \gg 1$ depending upon $b^+(X)$

instead.

proof is gluing of ASD connections on S^4

as dipicted earlier.

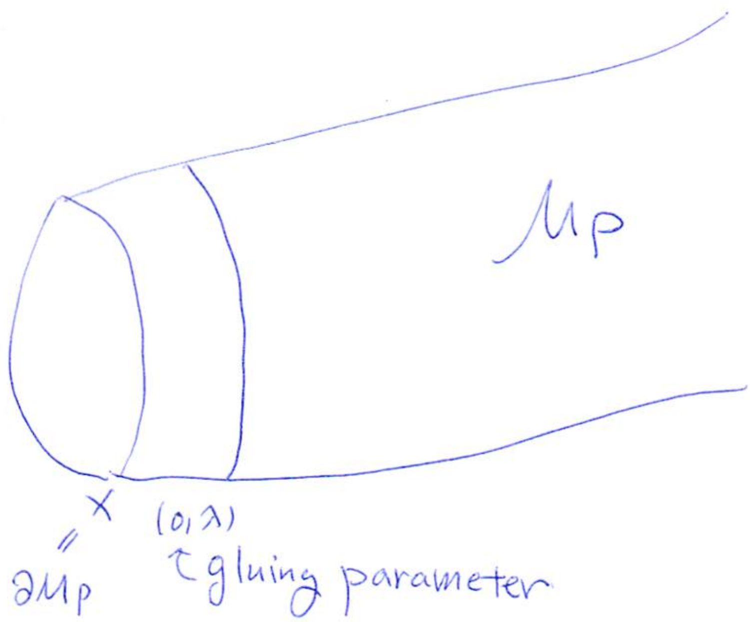
$c_2(P) = 1$

$\Rightarrow$ bubbling is at a point on X

if $b^+(X) = 0$ and $\pi_1(X) = \{1\}$, then

$$-\text{index} = 8c_2(P) - 3(1+b^+)$$
$$= 5$$

collar theorem coming from Taubes' theorem



§10 Donaldson's theorem

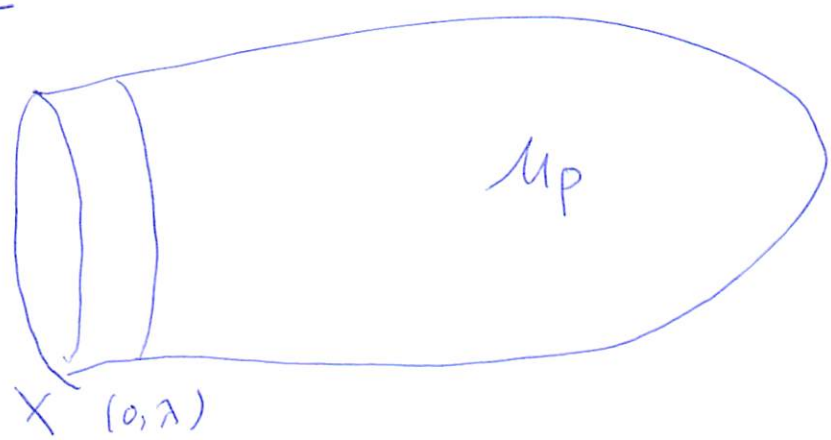
X : closed, oriented, smooth 4-manifold

$\pi_1(X) = \{1\}$, $b^+ = 0$

Consider $SU(2)$-bundle P with $c_2(P) = 1$

$\Rightarrow$ $-$index $=$ virtual dimension of $\mathcal{M}_P = 5$

Taubes



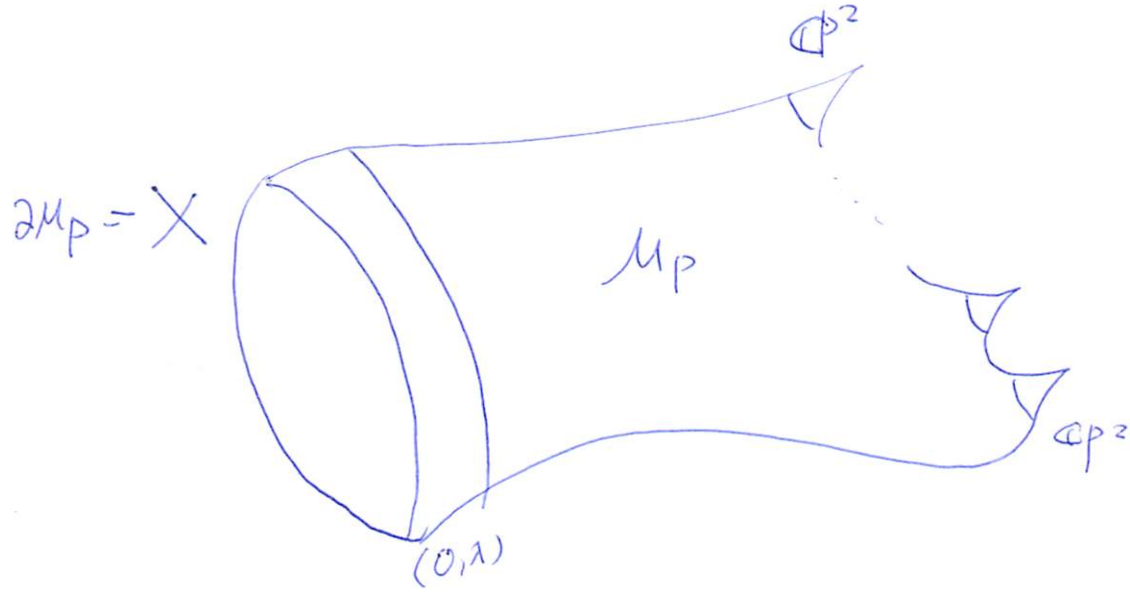
$\mathcal{M}_P$ is a smooth, oriented 5-dimensional manifold with $\partial \mathcal{M}_P = X$, except at reducible ASD connections for a generic choice of Riemannian metrics on X.

Recall in §6 (Lec 5)

the number of reducible connections is

$$m := \frac{1}{2} \left| \{ \alpha \in H^2(X, \mathbb{Z}) \mid Q(\alpha, \alpha) = 1 \} \right|$$

Locally at these points, $\mathcal{M}_P$ is a cone over $\mathbb{C}P^d$, where $d = 4c_2(P) - 2 = 2$



So X is oriented cobordant to $\coprod_m \overline{\mathbb{C}P^2}$

Thm (Donaldson)

Let X : closed oriented, simply-connected, smooth 4-manifold with negative (or positive) definite intersection form.

Then the intersection form can be diagonalised over $\mathbb{Z}$

cf. M. Freedman:

any unimodular symmetric form can be realized as the intersection form of a compact simply-connected topological 4-manifold.

Donaldson's theorem above gives a restriction on the intersection form of the smooth 4-manifolds with negative (or positive) definite intersection forms.

(also it can be used to prove the existence of exotic $\mathbb{R}^4$s.)